matical philosophers and logicians as Bertrand Russell[1] and A. N. Whitehead[2] as steps toward greater mathematical rigor. It must be emphasized, however, that the school of Weierstrass has not found universal recognition; there are modern champions of the infinitely small, chief among whom is the Italian mathematician Giuseppe Veronese. They insist that the Cantor continuum is not the only possible non-contradictory continuum and proceed to construct a higher and more involved, non-archimedean, continuum in which infinitely small distances are given. This is not the place for attempting a minute statement of the controversy between the two schools; the controversy, by the way, has no national aspect. There have been followers of Veronese in Germany (for instance, Stolz, Max Simon), and followers of Weierstrass and G. Cantor in Italy (for instance, Peano). So far as we have noticed, the Zeno arguments have not been studied and given explicit treatment on the basis of the Veronese continuum.[3] In America C. S. Peirce has adhered to the idea of infinitesimals in the declaration: "The illumination of the subject by a strict notation for the logic of relatives had shown me clearly and evidently that the idea of an infinitesimal involves no contradiction."[4] Apparently, before he had acquired familiarity with the writings of Dedekind and Georg Cantor, C. S. Peirce had firmly recognized that for infinite collections the axiom, that the whole is greater than its part, does not hold.

[To be continued]

ON NAPIER'S FUNDAMENTAL THEOREM RELATING TO RIGHT SPHERICAL TRIANGLES.

By ROBERT MORITZ, University of Washington.

In view of the recent celebration of the tercentenary of the publication of Napier's greatest work, the "Mirifici logarithmorum canonis descriptio," it is highly fitting that his rule for the circular parts should be rescued from the rubbish heap of mnemotechnics and be assigned its proper place as the most

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[1] See, for instance, his article in the *International Monthly*, Vol. 4, 1901, p. 84 and seq.

[2] A. N. Whitehead, *Introduction to Mathematics*, New York and London, 1911, pp. 156, 226–229.

[3] References to this controversy are as follows: G. Veronese, *Grundzüge der Geometrie von mehreren Dimensionen*, übersetzt v. A. Schepp, Leipzig, 1894, Anhang, p. 631–701; Max Simon, "Historische Bemerkungen über das Continuum, den Punkt und die Gerade Linie," *Atti del IV. Congresso Internazionale dei matematici*, Roma, 1908, pp. 385–390; G. Cantor's letter to Vivanti, *Rivista di mat.* V, 104–108; G. Cantor's letter to Peano, *Rivista di mat.* V, 108–109; G. Cantor, "Zur Begründung der Transfiniten Mengenlehre I," *Mathematische Annalen*, Vol. 46, 1895, page 500; Frederico Enriques, *Probleme der Wissenschaft*, 2. Teil, übersetzt von K. Grelling, Leipzig und Berlin, 1910, pp. 324–329. An able discussion of infinity, infinitesimals and the continuum is given by Josiah Royce, a philosopher familiar with mathematical thought, in his *The World and the Individual*, New York, 1900, pp. 505–560. See also G. Cantor, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitsch. für Philosophie u. Philosophische Kritik*, Vol. 91, Halle, 1887, p. 113; O. Stolz in *Mathematische Annalen*, Bd. XVIII, p. 699, also in *Berichte des naturw.-medizin. Vereins in Innsbruck*, Jahrgänge 1881–82 und 1884, also in *Vorlesungen über allgem. Arithm.*, Leipzig, 1. Theil, 1885, p. 205.

[4] C. S. Peirce, "The Law of Mind" in *The Monist*, Vol. 2, 1892, p. 537.

beautiful theorem in the whole field of elementary trigonometry. It is one of the strange vicissitudes of fortune that the elegant proof which was clearly indicated by Napier himself in the fourth chapter of the second book of the "descriptio" and rediscovered by Lambert[1] and Ellis[2] should nevertheless have remained generally unknown to writers on trigonometry in the nineteenth century and that even to this day the impression generally prevails that Napier's rules are nothing more than mnemonic devices whose utility as an instrument may well be questioned. Excepting two recent texts[3] one seeks in vain for any intimation that Napier's rules for the circular parts have any other than an inductive basis.

Thus as omniscient a writer as DeMorgan in his *Spherical Trigonometry* speaks of Napier's rules as "mnemonical formulas" and expresses his conviction that they "only create confusion instead of assisting the memory." Chauvenet (*Plane and Spherical Trigonometry*, 1891) after developing the ten formulas for the solution of right spherical triangles, says: "By putting these ten rules under a different form, Napier contrived to express them all in two rules, which, though artificial, are very generally employed as aids to the memory." In like tenor Newcomb (*Trigonometry*, 1893): "The six preceding formulæ, which may be found difficult to remember, have been included by Napier in two precepts of remarkable simplicity, and easily remembered" and the same view is reiterated in the recent work of Bôcher and Gaylord (*Trigonometry*, 1914) in the words "Formulas 1–10 may be collected into a very compact and convenient form by means of a rule formulated by John Napier. The student should prove that these rules are correct by applying them in succession to all five parts of the figure."

Nor is the impression that Napier's rules have no other than an inductive basis limited to writers of textbooks on trigonometry. Cajori in his *History of Mathematics* states that "Napier's Rule of circular parts is perhaps the happiest example of artificial memory that is known," thus putting this remarkable achievement in deduction on a par and in direct competition with the famous mnemonic hexameter of the logicians,

> "Barbara, Celarent, Darii, Ferioque prioris.
> Cesare, Camestres, Festino, Baroko secundae.
> Tertia Darapti, Disamis, Datisi, Felapton,
> Bokardo, Ferison habet. Quarta insuper addit
> Bramantip, Camenes, Dimaris, Fesapo, Fresison,"

of which Hamilton said "there are few human inventions which display a higher ingenuity."

Even as high and recent an authority as E. W. Hobson in his article on trigonometry in the eleventh edition of the Encyclopedia Britannica dismisses the whole matter with the words "Napier gave mnemonical rules for remembering" the right spherical triangle relations.

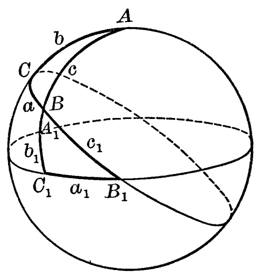
[1] *Beiträge zur Mathematik*, I (1765), p. 375 et seq.

[2] *The Mathematical and other Writings of Robert Leslie Ellis* (1863), p. 328 et seq.

[3] Leathem-Todhunter: *Spherical Trigonometry*, Macmillan (1907). Moritz: *Spherical Trigonometry*, Wiley and Sons (1914).

In view of this generally prevailing misconception, it may be timely to present in this place an elementary proof of Napier's theorem which will in a sense supplement Professor E. O. Lovett's prior note[1] on the same subject.

Let $\Delta = ABC$ be a right spherical triangle whose parts taken in counter-clockwise order beginning with a side adjacent to the right angle are a, B, c, A, b. With A as a pole construct the arc of a great circle intersecting the arcs CB produced and AB produced in B_1 and C_1 respectively. A second right triangle $\Delta_1 = A_1B_1C_1$ is thus formed whose parts a_1, B_1, c_1, A_1, b_1 are



$$a_1 = \bar{A},\ B_1 = \bar{b},\ c_1 = \bar{a},\ A_1 = B,\ b_1 = \bar{c},$$

where

$$\bar{A} = 90° - A, \qquad \bar{b} = 90° - b, \qquad \bar{a} = 90° - a, \text{ etc.}$$

Thus every triangle Δ whose circular parts are $a\ \bar{B}\ \bar{c}\ \bar{A}\ b$, leads to a second triangle Δ_1 whose circular parts are $\bar{A}\ b\ a\ \bar{B}\ \bar{c}$, and this likewise to a third Δ_2 whose circular parts are $\bar{B}\ \bar{c}\ \bar{A}\ b\ a$, and this again to a fourth Δ_3 whose circular parts are $b\ a\ \bar{B}\ \bar{c}\ \bar{A}$, and this in turn to a fifth Δ_4 whose circular parts are $\bar{c}\ \bar{A}\ b\ a\ \bar{B}$. The fifth triangle, Δ_4, leads to the first, Δ, thus completing the cycle.

The circular parts of these five triangles are thus shown to be the same for all. Furthermore, if we call the hypotenuse of a right triangle its middle part, this name may be applied to each of the circular parts, for each of the circular parts $a, \bar{B}, \bar{c}, \bar{A}, b$, is the hypotenuse of some one of the five triangles Δ, Δ_1, Δ_2, Δ_3, Δ_4. It follows that the relation between any middle part and its two adjacent (contiguous) parts applies to every middle part and its two adjacent parts, and again, that the relation between any middle part and its two opposite (non-contiguous) parts applies to every middle part and its two opposite parts.

Now from the first triangle ABC we find from the law of cosines (or otherwise) that $\cos c = \cos a \cos b$, that is $\sin \bar{c} = \cos a \cos b$, and on eliminating from this equation a and b by the law of sines (or otherwise) we obtain $\cos c = \cot A \cot B$, that is $\sin \bar{c} = \tan \bar{A} \tan \bar{B}$. This proves *Napier's Theorem: The sine of any circular part is equal to the product of the cosines of the opposite parts and to the product of tangents of the adjacent parts.*

[1] *Bulletin of the American Mathematical Society*, 1898, p. 552.